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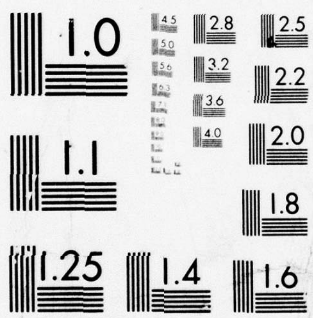

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February 1977

THE ANALYSIS OF PROPAGATION ON PERIODICALLY SLOTTED COAXIAL CABLE ABOVE LOSSY GROUND

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This technical report has been reviewed and approved for publication.

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After formulation of an integral equation and of the dispersion relation for a periodically slotted coaxial cable above and parallel to the air-lossy ground planar interface, the report addresses the evaluation of the exterior dyadic Green's function in a unit cell. The latter constitutes the heretofore not available in the literature kernel of the integral equation. The problem is reduced to an evaluation of the field of electric and magnetic multipole phased line sources above a lossy ground. The analysis of the various relevant integrals is carried out either in terms of an asymptotic expansion or rigorously by function theoretical methods.

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EVALUATION

1. This report is the Final Report on the contract. It covers research done on radiative transmission lines during the eleven-month period from 1 Jan to 30 Nov 1976. Radiative transmission lines are applicable to several radar and communication functions. The particular electromagnetic guiding structure analyzed in this report is a periodically slotted dielectric coaxial line above, parallel to, or buried in, a lossy ground. The ground modifies the propagation characteristics of the coaxial line, reducing the range at which radar targets can be detected. The objective of this research is to investigate theoretically the effects of a lossy ground on the transmission characteristics of a radiative transmission line. A second problem considered is the radar scattering by obstacles, such as vehicles or personnel, in the guided wave fields. The contractor carried out a rigorous mathematical analysis of the periodically slotted transmission line which relates its radiative properties to its physical configuration. These relations can be used as the basis for determining the effects of ground conductivity on the performance characteristics of radiative lines for guided wave radar applications.

2. The above work is of value since it provides basic knowledge which makes possible improved and optimized electromagnetic devices for USAF security systems, radar, and communications systems.

Walter Rotman

WALTER ROTMAN
Project Engineer

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Abstract

After formulation of an integral equation and of the dispersion relation for a periodically slotted coaxial cable above and parallel to the air-lossy ground planar interface, the report addresses the evaluation of the exterior dyadic Green's Function in a unit cell. The latter constitutes the heretofore not available in the literature kernel of the integral equation. The problem is reduced to an evaluation of the field of electric and magnetic multipole phased line sources above a lossy ground. The analysis of the various relevant integrals is carried out either in terms of an asymptotic expansion or rigorously by function theoretical methods.

1. Introduction

The influence of lossy ground on the propagation characteristics of periodically slotted coaxial cable is of interest in guided wave radar, or in intrusion prevention.

This study presents an approach to a systematic development of the formalism necessary for the derivation of the dispersion relation of a periodically slotted coaxial cable above and parallel to a flat lossy ground-air interface. Continuity of tangential fields across a slot in a unit cell yields an integral equation over the slot. The kernel of the integral equation is the sum of the exterior and the interior unit cell dyadic Green's functions, for the unperforated coaxial cable. This integral equation may be solved, e.g., via Galerkin's procedure which yields a set of homogeneous linear equations. Setting the determinant equal to zero yields the dispersion relation for the slotted cable. The exterior dyadic Green's Function is separated into that of the cable in free space plus a correction which contains the effect of the ground. As shown in Appendix A (see also [1]), the exterior dyadic Green's function for the slotted cable in free space may be derived from two axial Hertz potentials, which are expanded in terms of axial and angular harmonics. Each combination of a spatial and an angular harmonic is identical in form with a Hertz potential of an electric or magnetic current phased line source located on the axis of the coaxial cable. Thus, if one neglects multiple interactions, the effect of the ground is obtained as a superposition of back scattered fields excited by the various phased line sources. Hence, the evaluation of the exterior dyadic Green's function is reduced to the determination of fields excited by an electric or a magnetic phased line source above and parallel to a lossy ground. The case of a phased electric line

source was treated in [2]. Reference [2], however, limits its formulae to the case when the dielectric constant ϵ of the ground is sufficiently large, so that the propagation constant of the ground may be replaced by $k\sqrt{\epsilon}$. This procedure restricts the distance of the line source from the interface to be sufficiently large, so that the higher order terms arising from the expansion of the ground propagation constant may be neglected. However, in this case, the effect of ground would, in general, be a small perturbation and the propagation characteristics of the cable would be practically independent on the presence of ground.

To amend the situation for the TE modes this report carries out the calculations to higher order in terms of an asymptotic multipole series. For the TM case the procedure of Reference [2] modifies the location of Sommerfeld poles, which in certain cases, particularly for not very large values of $|\epsilon|$, may give rise to discrepancies. Furthermore, the validity of the analytic continuation based on values of the integral at four different points (Appendix A of [2]) is questionable. On the other hand the method of evaluation given in this report has a rigorous basis; no approximation in the location of the poles is made here and, a rigorous result is obtained via a contour deformation, which explicitly separates the pole contribution from the quasistatic term plus a rapidly convergent correction.

The additional advantage of the separation of the pole contribution is that it permits a simple continuous tracking of this contribution in a numerical solution of the dispersion relation, when the propagation constant β of the cable becomes complex. In contrast, the function that appears in [2] may become discontinuous, a feature not admissible in a dispersion relation.

The organization of this report is as follows:

Section 2 discusses the integral equation for a slotted coaxial cable, and the resulting form of the dispersion relation which is applicable for both the slow and the fast wave regions.

Section 3 addresses the derivation for the exterior dyadic Green's function which constitutes the key ingredient in the kernel of the integral equation. The problem is reduced to the evaluation of certain canonical integrals.

Section 4 presents an asymptotic evaluation of the TE contribution.

Section 5 addresses the evaluation of the TM contributions in terms of Sommerfeld wave residue an explicit quasistatic part and two rapidly convergent integrals. The explicit residue contribution permits a simple analytic continuation of the functions appearing in the dispersion relation. Appendix A, after summary of radial line formalism appropriate to unit cell in a cylindrical structure, derives the exterior, unit cell, dyadic Green's function for the coaxial cable in free space. Appendix B shows that the electromagnetic fields due to phased line source parallel to an interface between two homogeneous dielectric half spaces is derived from two axial Hertz potentials. The expressions for the potentials are derived. Appendix C discusses the location of Sommerfeld poles arising in the TM contribution to the field excited by line sources in the presence of lossy ground.

2. Analysis of the Dispersion Relation for a Periodically Slotted Coaxial Cable Above Lossy Ground.

Figure (1) shows the geometry of a dielectric-loaded, periodically slotted coaxial cable with a zero thickness shield. A dielectric jacket is not included, but its presence could be easily incorporated.

Figure (2) shows such cable above, and parallel to, the planar interface between air and a losssy ground. A unit cell of the structure is contained between two parallel planes perpendicular to the cable axis and a distance d apart. The dispersion relation for the periodically slotted cable is obtained with the help of an integral equation expressing the continuity of the tangential magnetic and electric fields across a typical slot in a unit cell. The kernel of the integral equation in a sum of two unit cell dyadic Green's functions $\underline{\underline{G}}_E$ and $\underline{\underline{G}}_I$. The former (latter) yields the magnetic field due to an elemental magnetic current placed in the slot location in the exterior (the interior) of the unperforated structure, in a unit cell. That is to say, $\underline{\underline{G}}_E(\underline{r}, \underline{r}' \beta)$ is the unit cell dyadic exterior G. F. for the unperforated, perfectly conducting cylinder and $\underline{\underline{G}}_I$ is such a G.F. for the interior unperforated coaxial cable. Both $\underline{\underline{G}}_E(\underline{r}, \underline{r}')$ and $\underline{\underline{G}}_I(r, r')$ satisfy the Floquet conditions which implicitly include all interactions between neighboring slots :

$$\underline{\underline{G}}_E(\underline{r}+\underline{x}_o d, \underline{r}'; \beta) = e^{-j\beta d} \underline{\underline{G}}_E(\underline{r}, \underline{r}'; \beta)$$

$$\underline{\underline{G}}_I(\underline{r}+\underline{x}_o d, \underline{r}'; \beta) = e^{-j\beta d} \underline{\underline{G}}_I(\underline{r}, \underline{r}', \beta) \quad . \tag{1}$$

The integral equation reads

$$\int_{slot} \left[\underline{\underline{G}}_E(\underline{r}, \underline{r}'; \beta) + \underline{\underline{G}}_I(r, r'; \beta) \right] \cdot \underline{E} x \underline{\rho}'_o \, dS' = 0 \, , \ \underline{r} \text{ in the slot} \, , \tag{2}$$

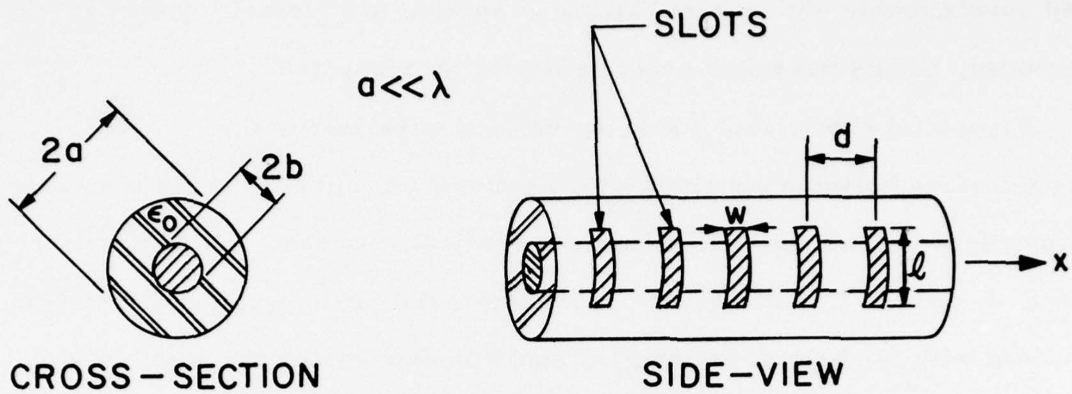

Fig. 1 Periodically Slotted Coaxial Cable Geometry

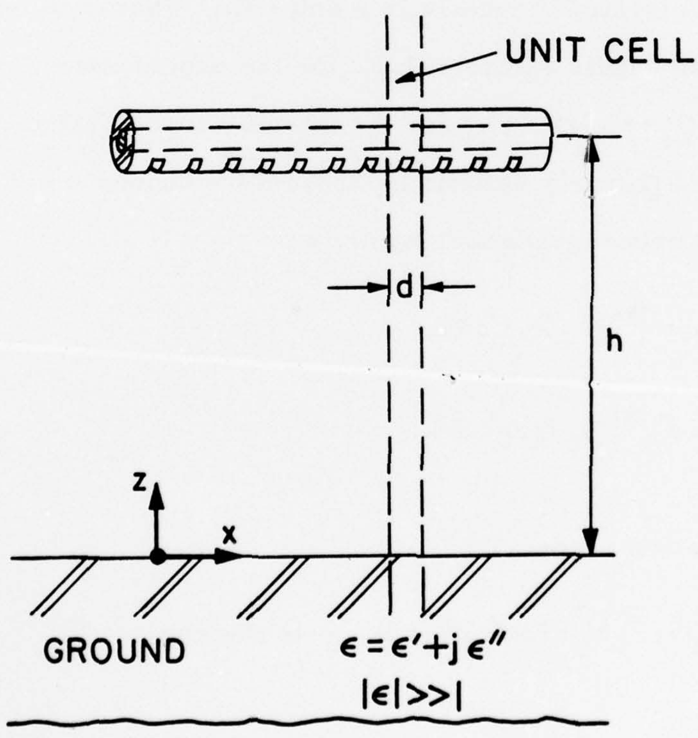

Fig. 2 Slotted Cable Above Ground

where $\underline{E}(\underline{r}')$ is the unknown slot electric field and $\underline{\rho}'_o$ is the radial unit vector at $\underline{r}'$.

Application of Galerkin's procedure, i. e. an expansion of the slot field in terms of a linearly independent, preferably orthnormal, finite set of basis functions, and the requirement that the resultant vector function (the integral) be orthogonal to each member of the chosen basis, yields a homogeneous linear system of equations for the unknown expansion coefficients. The dispersion relation is obtained by setting the system determinant to zero. In particular, assuming only a sinusoidal distribution $\underline{e}_o$ in the slot, one finds an approximate dispersion relation

$$\iint_{\text{slot slot}} \underline{e}_o(\underline{r}) \times \underline{\rho}_o \cdot \left[\underline{\underline{G}}_E(\underline{r},\underline{r}';\beta) + \underline{\underline{G}}_I(\underline{r},\underline{r}';\beta)\right] \cdot \underline{e}_o(\underline{r}') \times \underline{\rho}'_o \, dS \, dS' = 0 \qquad (3)$$

which is to be solved for the unknown propagation constant β as a function of frequency. It is seen that the basic ingredients in (3) are $\underline{\underline{G}}_E$ and $\underline{\underline{G}}_I$. The expression for G_I are obtained in a standard fashion along the lines of Appendix A. The major problem in this study is the determination of $\underline{\underline{G}}_E$.

3. The Exterior Unit Cell Dyadic Green's Function $\underline{\underline{G}}_E(\underline{r}, \underline{r}'; \beta)$

For the unperforated circular cylinder the unit cell exterior dyadic Green's function $\underline{\underline{G}}_E$ is represented as a sum of two contributions

$$\underline{\underline{G}}_E = \underline{\underline{G}}_{EF} + \underline{\underline{G}}_{EG} \tag{4}$$

where $\underline{\underline{G}}_{EF}$ is the unit cell exterior dyadic Green's Function for the unperforated cylinder in free space and $\underline{\underline{G}}_{EG}$ represents the effect of the ground and the multiple interactions between the cylinder and lossy earth interface.

An approximation for $\underline{\underline{G}}_{EG}$ is now made which considers only a single ground reflection. This approximation should be adequate, except when the cable is very close to the ground.

3a) The $\underline{\underline{G}}_{EF}$

As shown in Appendix A, $\underline{\underline{G}}_{EF}$ is derived from two axial Hertz potentials π' and π'' which, for the slow wave case to be discussed in this report, are of the form

$$\begin{aligned} \pi'(\underline{r}, \underline{r}'; \beta) &= \sum_{n,m=-\infty}^{\infty} A'_{nm} K_m(\tau_n \rho) e^{-jm\varphi} e^{-j\beta_n x} \\ \pi''(r, r'; \beta) &= \sum_{n,m=-\infty} A''_{nm} K_m(\tau_n \rho) e^{-jm\varphi} e^{-j\beta_n x} \qquad -\frac{d}{2} < x < d/2 \end{aligned} \tag{5}$$

where $\beta_n = \beta + \frac{2\pi}{d} n$ and $\tau_n = \sqrt{\beta_n^2 - k^2}$ are the space harmonic axial and transverse propagation constants.

In view of Floquet's Theorem, the validity of (5) extends also for $-\infty < x < \infty$. Thus, a typical term in (5) represents an axial electric or magnetic Hertz potential

$$A_m K_m(\tau\rho)e^{-jm\varphi} e^{-j\beta x} \tag{6}$$

of a multipole slow-wave ($\beta > k$) phased line source located on the cylinder axis. In view of the linearity of Maxwell's equations it suffices to solve for the disturbance due to lossy ground in the case of a monopole, i.e. for an isotropic electric and magnetic current phased line source. This is because

$$K_m(\tau\rho)e^{-jm\varphi} = \frac{(-1)^m}{\tau^m}\left[e^{-j\varphi}\left(\frac{\partial}{\partial\rho} - j\frac{\partial}{\rho\partial\varphi}\right)\right]^m\left[K_o(\tau\rho)\right] = L^m K_o(\tau\rho) =$$

$$\frac{(-1)^m}{\tau^m}\left(\frac{\partial}{\partial z} - j\frac{\partial}{\partial y}\right)^m\left[K_o(\tau\rho)\right] \quad . \tag{7}$$

Therefore, the expressions for the fields excited by an mth multipole line source above ground may be generated by the application of the linear differential operator $(L)^m$ to the fields due to an appropriate isotropic phased line source above, and parallel to, a lossy ground interface.

3b) Basic Integrals for $\underline{\underline{G}}_E$.

In Appendix B it is shown that the fields of an electric or a magnetic line source above, and parallel to, a dielectric half space may be derived from two Hertz potentials. From (B 14, 15, 26 and 27) one observes that the basic types of integrals to be evaluated in order to determine $\underline{\underline{G}}_E$ are

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{-j\eta y}}{(\varkappa_1+\varkappa_2)} \frac{e^{-j\varkappa_1(z+z')}}{\varkappa_1} d\eta \qquad \varkappa_1 = \sqrt{k^2-\beta^2-\eta^2}\,, \quad \mathrm{Im}\,\varkappa_1 < 0$$

$$\varkappa_2 = \sqrt{k^2\epsilon-\beta^2-\eta^2}\,, \quad \mathrm{Im}\,\varkappa_2 < 0 \tag{8}$$

$$I_2 = \int_{-\infty}^{\infty} \frac{e^{-j\eta y}\, e^{-j\varkappa_1(z+z')}}{(\varkappa_2+\epsilon\varkappa_1)\,\varkappa_1} d\eta \quad .$$

Other integrals that appear are obtained via the operations $\frac{\partial}{\partial y}$ or $\frac{\partial}{\partial z}$ applied to I_1 or I_2. As indicated, we shall restrict our considerations to the slow wave case, i.e., such that for $\beta = k\sqrt{\epsilon_c}$ (ϵ_c being the dielectric constant loading of the coaxial cable), the spacing d/λ is sufficiently small so that for all n, $\beta_n^2 > k^2$. At higher frequencies the cable will become leaky when one of the space harmonics becomes fast. The expressions for a leaky cable may be derived via minor modifications of the results given in this report. I_1 and I_2 may be further simplified as follows:

$$I_1 = \frac{1}{k^2(\epsilon-1)} \int_{-\infty}^{\infty} \frac{(\varkappa_2-\varkappa_1)}{\varkappa_1} e^{-j\eta y} e^{-j\varkappa_1(z+z')} d\eta = \frac{\pi}{jk^2(\epsilon-1)} \frac{\partial}{\partial z} H_o^{(2)}\left(\sqrt{k^2-\beta^2}\sqrt{y^2+(z+z')^2}\right) +$$

$$+ \frac{1}{k^2(\epsilon-1)} \int_{-\infty}^{\infty} \frac{\varkappa_2}{\varkappa_1} e^{-j\eta y} e^{-j\varkappa_1 Z} d\eta \tag{9}$$

where $Z = z+z'$ and $H_o^{(2)}$ is the Hankel function of the zeroth order and 2nd kind.

The integral I_2 may be rewritten as:

$$I_2 = \frac{1}{\epsilon^2-1} \int_{-\infty}^{\infty} \frac{(\varkappa_2-\epsilon\varkappa_1)}{\eta^2-a^2} \frac{e^{-j\eta y}\, e^{-j\varkappa_1 Z}}{\varkappa_1} d\eta \quad , \tag{10}$$

where $$a^2 = k^2 \frac{\epsilon}{1+\epsilon} - \beta^2 \tag{11}$$

with $$a = \sqrt{k^2 \frac{\epsilon}{1+\epsilon} - \beta^2} \quad ; \quad \text{Im } a < 0 \ . \tag{12}$$

The poles at $\eta_p = \pm a$ are the so-called Sommerfeld poles.

The basic integrals to be evaluated are therefore

$$F(y, Z) = \int_{-\infty}^{\infty} \varkappa_2 \frac{e^{-j\eta y} e^{-j\varkappa_1 Z}}{\varkappa_1} d\eta \tag{13}$$

$$G(y, Z) = \int_{-\infty}^{\infty} \frac{e^{-j\eta y} e^{-j\varkappa_1 Z}}{(\eta^2 - a^2)\varkappa_1} d\eta$$

$$H(y_1 Z) = \int_{-\infty}^{\infty} \frac{\varkappa_2 e^{-j\eta y} e^{-j\varkappa_1 Z}}{(\eta^2 - a^2)\varkappa_1} d\eta \ . \tag{15}$$

$F(y, z)$ will be evaluated as an asymptotic series in terms of the canonical integral

$$\int_{-\infty}^{\infty} \frac{e^{-j\eta y - j\varkappa_1 Z}}{\varkappa_1} d\eta = \pi H_o^{(2)}\left(\sqrt{k^2 - \beta^2} \sqrt{y^2 + Z^2}\right) \tag{16}$$

This expansion is valid for sufficiently large distances and will be described in Section 4 .

The canonical integral $G(y, Z)$ is evaluated exactly, and $H(y,Z)$ is in turn expanded in an asymptotic series in terms of $G(y,Z)$ and its derivatives. The integral $F(y, Z)$ represents the TE mode contribution ,

while G and H , which contain the Sommerfeld poles, represent the TM mode contributions (with respect to z).

4) Image Series Representation for the TE Contribution F(y, Z)

We are considering here the slow wave case, $\beta > k$ and define now the following quantities:

$$\gamma_1 = \sqrt{\beta^2 - k^2} > 0, \quad \varkappa_1 = -j\sqrt{\gamma_1^2 + \eta^2} = -j\gamma , \qquad \mathrm{Re}\, \gamma > 0 \ , \tag{17}$$

$$\gamma_2^2 = k^2(\epsilon - 1) \ , \quad \varkappa_2 = \sqrt{\gamma_2^2 - \gamma^2} \ , \quad \mathrm{Im}\, \varkappa_2 < 0 \ , \tag{19}$$

$$\gamma_2 = k\sqrt{\epsilon - 1} \ , \quad \mathrm{Im}\, \gamma_2 < 0 \ . \tag{19}$$

F(y, Z) may now be written:

$$F(y, Z) = j \int_{-\infty}^{\infty} \varkappa_2 \frac{e^{-j\eta y} e^{-\gamma Z}}{\gamma} \, d\eta \tag{20}$$

When $\varkappa_2 = 1$ the integral in (20) represents the modified Bessel function $2\, K_0(\gamma_1 \rho)$, $\rho = \sqrt{y^2 + Z^2}$. In view of the factor $e^{-\gamma Z}$, the major contribution to the integral arises from the neighborhood of $\eta = 0$. As a result, when $|\epsilon|$ is large, $\sqrt{\gamma_2^2 - \gamma^2}$ in (20) may be expanded in a Taylor series about $\gamma = 0$:

$$\sqrt{\gamma_2^2 - \gamma^2} = \gamma_2 \left[1 - \frac{1}{2} \frac{\gamma^2}{\gamma_2^2} - \frac{\gamma^4}{8\gamma_2^4} - \ldots \right], \ \mathrm{Im}\, \gamma_2 < 0 \tag{21}$$

and the resultant series is integrated term by term. The integration is facilitated by the presence of powers of γ , whereby $\gamma^2 \to \frac{\partial^2}{\partial Z^2}$ acting on the basic integral.

Thus,

$$F(y,z) = 2\gamma_2\left[K_o(\gamma_1\rho) - \frac{1}{2\gamma_2^2}\frac{\partial^2}{\partial Z^2}K_o(\gamma_1\rho) - \frac{1}{8\gamma_2^4}\frac{\partial^4}{\partial Z^4}K_o(\gamma_1\rho) - \ldots\right] \quad . \tag{22}$$

The series in (22) is a multipole expansion, the first term representing the monopole, the next a quadrupole etc.

As a result,

$$F(y,Z) \sim 2\gamma_2\Big\{K_o(\gamma_1\rho) - \frac{1}{2\gamma_2^2}\Big[K_o(\gamma_1\rho)\frac{\gamma_1^2 Z^2}{\rho^2} - K_1(\gamma_1\rho)\Big(\frac{\gamma_1}{\rho} - \frac{2\gamma_1 Z^2}{\rho^3}\Big)\Big] -$$

$$- \frac{1}{8\gamma_2^4}\Big[K_o(\gamma_1\rho)\Big(\frac{3\gamma_1^2}{\rho^2} - \frac{24\gamma_1^2 Z^2}{\rho^4} + \frac{\gamma_1^4 Z^4}{\rho^4} + \frac{24\gamma_1^2 Z^4}{\rho^6}\Big) +$$

$$+ K_1(\gamma_1\rho)\Big[\frac{6\gamma_1}{\rho^3} - \frac{6\gamma_1^3 Z^2}{\rho^3} - \frac{48\gamma_1 Z^2}{\rho^5} + \frac{8\gamma_1^3 Z^4}{\rho^5} + \frac{48\gamma_1 Z^4}{\rho^7}\Big]\Big]\Big\} + \ldots \tag{23}$$

$$= 2\gamma_2\Big\{K_o(\gamma_1\rho) - K_o(\gamma_1\rho)\frac{1}{2}\Big(\frac{\gamma_1}{\gamma_2}\Big)^2\Big(1 - \frac{y^2}{\rho^2}\Big) - \frac{(\gamma_1\rho)K_1(\gamma_1\rho)}{2(\gamma_2\rho)^2}\Big(1 - \frac{2y^2}{\rho^2}\Big)$$

$$- K_o(\gamma_1\rho)\Big(\frac{\gamma_1}{\gamma_2}\Big)^2\Big[(\gamma_2\rho)^{-2}\Big(\frac{3}{8} - 3\frac{y^2}{\rho^2}\Big(1 - \frac{y^2}{\rho^2}\Big) + \frac{1}{8}\Big(\frac{\gamma_1}{\gamma_2}\Big)^2\Big(1 - \frac{y^2}{\rho^2}\Big)^2\Big]$$

$$- (\gamma_1\rho)K_1(\gamma_1\rho)(\gamma_2\rho)^{-2}\Big[\frac{3}{4}(\gamma_2\rho)^{-2} - \frac{3}{4}\Big(\frac{\gamma_1}{\gamma_2}\Big)^2\Big(1 - \frac{y^2}{\rho^2}\Big) - 6(\gamma_2\rho)^{-2}\Big(1 - \frac{y^2}{\rho^2}\Big) +$$

$$+ \Big(\frac{\gamma_1}{\gamma_2}\Big)^2\Big(1 - \frac{y^2}{\rho^2}\Big)^2 + 6(\gamma_2\rho)^{-2}\Big(1 - \frac{y^2}{\rho^2}\Big)^2\Big]\Big\} + 0\Big[\Big(\frac{\gamma_1}{\gamma_2}\Big)^p(\gamma_2\rho)^{-q}\Big]_{p+q=6} \tag{24}$$

When the cable is not too close to the ground surface, then, for observation points on the cable (or on a typical slot) $y/\rho << 1$, and one may neglect in (24) all powers of $\left(\frac{y}{\rho}\right)^2$ and higher, or retain only the y independent terms plus $0\left(\frac{y}{\rho}\right)^2$.

For $y=0$ one has

$$F(0,Z) \sim 2\gamma_2 \Big\{ K_o(\gamma_1 Z) - K_o(\gamma_1 Z)\frac{1}{2}\left(\frac{\gamma_1}{\gamma_2}\right)^2 - \frac{(\gamma_1 Z)K_1(\gamma_1 Z)}{2(\gamma_2 Z)^2}$$

$$- \left(\frac{K_o(\gamma_1 Z)}{2}\left(\frac{\gamma_1}{\gamma_2}\right)^2 + (\gamma_1 Z)K_1(\gamma_1 Z)(\gamma_2 Z)^{-2} \right)\left(\frac{3}{4}(\gamma_2 Z)^{-2} + \frac{1}{4}\left(\frac{\gamma_1}{\gamma_2}\right)^2 \right) + 0\left(\frac{\gamma_1}{\gamma_2}\right)^p(\gamma_2 Z)^{-q}$$

$$= 2\gamma_2 \Big\{ K_o(\gamma_1 Z) - \left[\frac{K_o(\gamma_1 Z)}{2}\left(\frac{\gamma_1}{\gamma_2}\right)^2 + \frac{(\gamma_1 Z)K_1(\gamma_1 Z)}{2(\gamma_2 Z)^2} \right]\left(1 + \frac{3}{4}(\gamma_2 Z)^{-2} + \frac{1}{4}\left(\frac{\gamma_1}{\gamma_2}\right)^2\right)\Big\} + 0\left(\frac{\gamma_1}{\gamma_2}\right)^p(\gamma_2 Z)^{-q} \quad p+q=6 \,. \tag{25}$$

It is seen that the asymptotic expansion (25) contains two small parameters $\left(\frac{\gamma_1}{\gamma_2}\right)^2$ and $(\gamma_2 Z)^{-2}$.

Example:

For $|\epsilon-1| = 10$, $\frac{\beta}{k} = 1.6$, $Z = z+z' = 0.2\lambda$

$$\gamma_1 Z = 1.57, \quad |\gamma_2 Z|^{-2} = 0.06\,, \quad \frac{1}{2}\left|\frac{\gamma_1}{\gamma_2}\right|^2 = 0.07 \,.$$

Hence,

$$\frac{\left|\frac{K_o(\gamma_1 Z)}{2}\left(\frac{\gamma_1}{\gamma_2}\right)^2\right| + \left|\frac{(\gamma_1 Z)K_1(\gamma_1 Z)}{2(\gamma_2 Z)^2}\right|}{K_o(\gamma_1 Z)} \approx 0.13$$

If one neglects the 3rd order term, when its magnitude is say 10% of the second order term, i. e., when

$$\left| \frac{3}{4}(\gamma_2 Z)^{-2} + \frac{1}{4}\left(\frac{\gamma_1 Z}{\gamma_2 Z}\right)^2 \right| \leq 0.1 \quad ,$$

then for $|\epsilon-1| = 10$ and $\frac{\beta}{k} = 1.6$,

$\frac{z+z'}{\lambda}$ must exceed 0.16. This is the case when, for $z=z'$,

$\frac{z'}{\lambda} \geq 0.08$, i. e., the height of the cable should be greater than approx. 0.1λ .

Thus, for given $\beta/k = \frac{c}{v_p}$ of the unperforated coaxial line and for given electrical properties of the lossy ground, the validity of the asymptotic expansion of F(y, Z) (24) or (25) is limited to distances z' not too close to the ground. Formulae (24) or (25) permit, in each case, to assess the validity of the expansion i. e., of the minimum distance of the line source (cable axis to ground) for a given value of $\frac{\beta}{k}$ and ϵ.

5) Evaluation of the TM Contribution G(y, Z)

Since, for the TM polarization, the air-lossy ground interface supports a Sommerfeld surface wave, it is clear that an image representation of G(y, Z) will not be useful.

Therefore, one has to resort to a type of evaluation which explicitly exhibits the surface wave contribution. Again we consider here only the slow wave case, for which

$$-jG(y, Z) = \int_{-\infty}^{\infty} \frac{e^{\sqrt{\gamma_1^2+y^2}\,Z}\; e^{-j\eta y}}{(\eta^2-a^2)\sqrt{\gamma_1^2+\eta^2}}\, d\eta = \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\gamma_1^2+\eta^2}\,Z}}{(\eta^2-a^2)\sqrt{\gamma_1^2+\eta^2}} \left(1-j\eta y-\frac{\eta^2 y^2}{2}+\dots\right) d\eta. \tag{26}$$

Thus, since the odd powers of y do not contribute

$$G(y, Z) = G(0, Z) + \left[\frac{\gamma_1^2}{2} G(0, Z) - \frac{\partial^2}{\partial Z^2} G(0, Z)\right] y^2 + 0(y^4) \quad , \tag{27}$$

where

$$G(0, Z) = G(Z) = \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\gamma_1^2 + \eta^2}\, Z}}{(\eta^2 - a^2)\sqrt{\gamma_1^2 + \eta^2}}\, d\eta \ . \tag{28}$$

It is therefore sufficient to evaluate $G(Z)$, whereupon $G(y, Z)$ follows via (27).

For the purpose of contour deformation that follows, it is necessary to discuss the properties of singularities in the complex η plane of the integrand in (28).

5a) The Top Sheet of $\sqrt{\gamma_1^2 + \eta^2}$ in the Complex η Plane.

We have

$$a^2 = k^2 \frac{\varepsilon}{1+\varepsilon} - \beta^2 \quad , \quad \text{Im}\, a < 0 \ \text{(for definiteness)} . \tag{29}$$

$$\text{Re} \sqrt{\gamma_1^2 + \beta^2} > 0 \quad , \quad \text{on the entire top sheet} \tag{30}$$

$$\beta = \beta_r + j\beta i \quad , \ \beta_r > 0, \ \beta_i < 0 \ \ \text{(for propagation in the +x direction with a simultaneous attenuation in the direction of propagation)} \tag{31}$$

$$\varepsilon = \varepsilon_r + j\varepsilon_i \quad , \quad \varepsilon_r > 0 \ , \ \varepsilon_i < 0 \tag{32}$$

$$\gamma_1 = \sqrt{\beta^2 - k^2} = \sqrt{\beta_r^2 - \beta_i^2 - k^2 + 2j\beta_r \beta_i} \quad . \tag{33}$$

Si nce, on the top sheet $\text{Re}\, \gamma_1 = \left(\text{Re} \sqrt{\gamma_1^2 + \eta^2}\right) > 0$,

and since $\beta_r \beta_i < 0$, one has the following two possibilities for γ_1^2 as shown in Fig. 3.

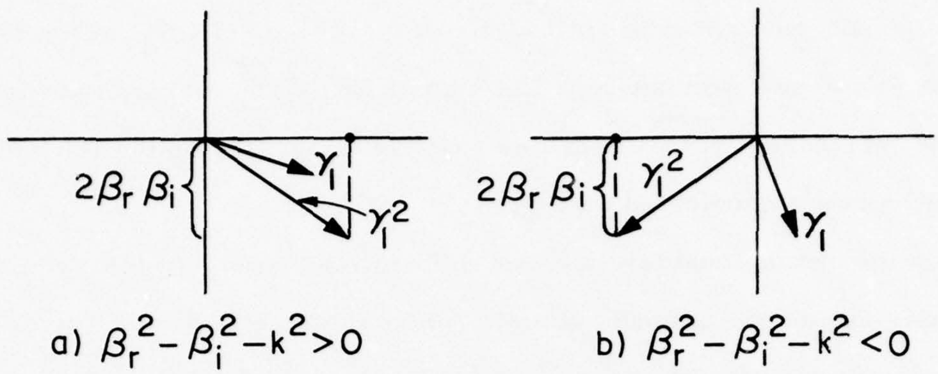

Fig. 3. Wavenumber γ_1, for Re $\gamma_1 > 0$.

It is seen that in either case Im $\gamma_1 < 0$.

For a slow wave Re $\beta > k$, one has therefore the following situation in the complex η plane (Fig. 4)

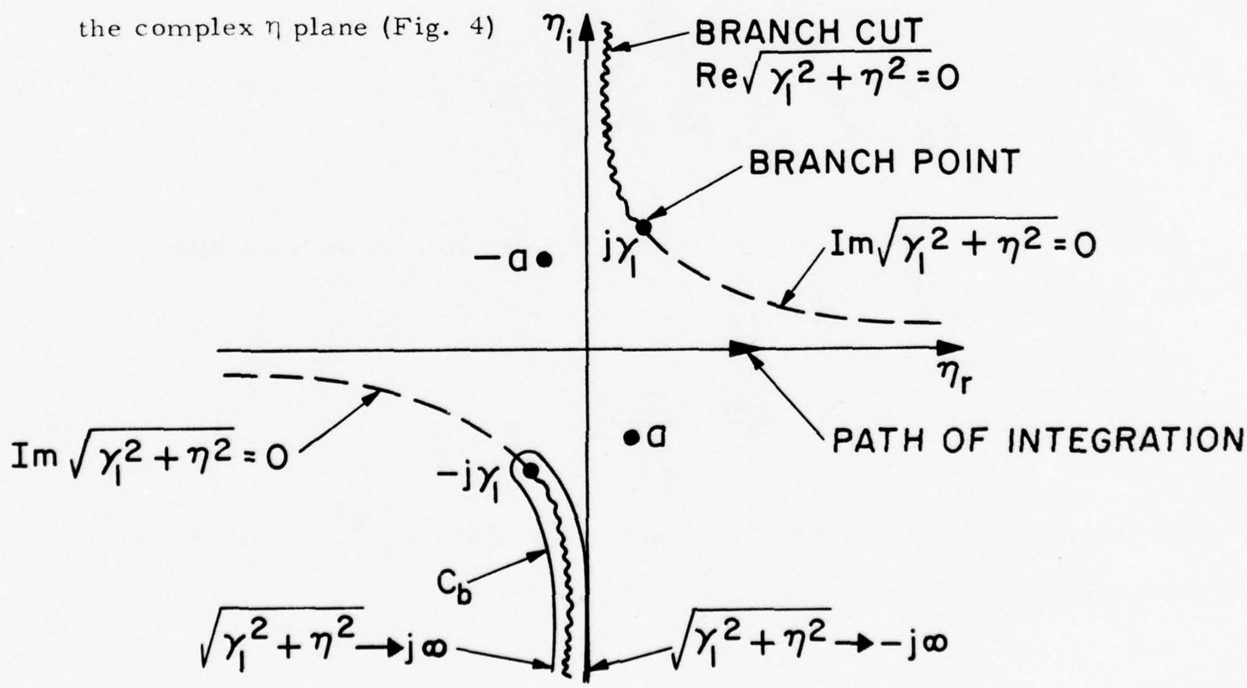

Fig. 4. Top Sheet Re $\sqrt{\gamma_1^2 + \eta^2} > 0$ in the complex η plane.

Along the branch cuts $\sqrt{\gamma_1^2+\eta^2}$ is purely imaginary. To determine the proper sign on either side of the lower branch cut we observe that, on the top sheet, the sign of $\mathrm{Im}\sqrt{\gamma_1^2+\eta^2}$ on the right bank of the lower branch-cut is that of $\mathrm{Im}\sqrt{\gamma_1^2+\eta^2}$ at $\eta = 0$ i.e. of $\mathrm{Im}\gamma_1$ which is negative as shown previously. The sign of $\mathrm{Im}\sqrt{\gamma_1^2+\eta^2}$ changes across the cut and therefore $\sqrt{\gamma_1^2+\eta^2}$ becomes positive imaginary on the left side of the lower cut, as indicated on Fig. 4 .

With the above considerations, we deform the contour in (28) from the real axis around the branch cut and capture the pole at $\eta = a$. The deformation is permissible because on the entire top sheet $\mathrm{Re}\sqrt{\gamma_1^2+\eta^2} > 0$. (Actually, since the term $e^{-j\eta y}$ has been removed from the integrand in G, the contour could be just as well deformed about the upper branch cut in Fig. 4.

The result of the contour deformation is:

$$G(Z) = -2\pi j \frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{2a\sqrt{\gamma_1^2+a^2}} + \int_{C_b} \frac{e^{-\sqrt{\gamma_1^2+\eta^2}\,Z}}{(\eta^2-a^2)\sqrt{\gamma_1^2+\eta^2}}\, d\eta \; . \tag{34}$$

As shown in Appendix C, the poles at $\eta = \pm a$ are located on the proper sheet, i.e. $\mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0$.

Changing variable via

$$\sqrt{\gamma_1^2+\eta^2} = \zeta \;, \text{ i.e. } d\eta = \frac{\zeta d\zeta}{\eta} = \frac{\zeta d\zeta}{\sqrt{\zeta^2-\gamma^2}} \;, \tag{35}$$

and remembering that $\mathrm{Im}\,\eta < 0$ on the cut, so that $\mathrm{Im}\sqrt{\zeta^2-\gamma_1^2} < 0$ on the contour C_b, one has

$$G(Z) = -\frac{j\pi}{a}\frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} + \int_{j\infty}^{-j\infty}\frac{e^{-j\zeta Z}\,d\zeta}{(\zeta^2-\gamma_1^2-a^2)\sqrt{\zeta^2-\gamma_1^2}}, \quad \mathrm{Im}\sqrt{\zeta^2-\gamma_1^2} < 0 . \tag{36}$$

Upon substitution $\zeta = j\xi$

$$(\text{or } \sqrt{\gamma_1^2+\eta^2} = j\xi) \tag{37}$$

one finds

$$G(Z) = \frac{-j\pi}{a}\frac{e^{-\sqrt{\gamma_1^2+a}\,Z}}{\sqrt{\gamma_1+a}} - j\int_{\infty}^{-\infty}\frac{e^{-j\zeta Z}\,d\zeta}{(\xi^2+\gamma_1^2+a^2)\sqrt{-(\xi^2+\gamma_1^2)}} =$$

$$= \frac{-j\pi}{a}\,\frac{e^{-\sqrt{\gamma_1^2+a}\,Z}}{\sqrt{\gamma_1^2+a^2}} - \int_{-\infty}^{\infty}\frac{e^{-j\xi Z}\,d\xi}{(\xi^2+\gamma_1^2+a^2)\sqrt{\xi^2+\gamma_1^2}}, \quad \begin{array}{l} \mathrm{Re}\sqrt{\xi^2+\gamma_1^2} > 0 \\ \\ \mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0 . \end{array} \tag{38}$$

Since $\xi = -j\sqrt{\gamma_1^2+\eta^2}$, the proper sheet $\mathrm{Re}\sqrt{\gamma_1^2+\eta^2} > 0$ maps into the lower half ξ plane $\mathrm{Im}\xi < 0$. The upper half ξ plane corresponds to the improper sheet of the $\sqrt{\gamma_1^2+\eta^2}$. Let the pole of the integrand at $\xi_{p1} = -j\sqrt{\gamma_1^2+a^2}$ have $\mathrm{Im}\xi_p < 0$ i.e. let $\mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0$. The other pole is then $\xi_{p2} = j\sqrt{\gamma_1^2+a^2}$ with $\mathrm{Re}\sqrt{\gamma_1^2+a^2} < 0$.

In this fashion the resulting contour of integration has been again brought to the real axis, and therefore the integral, which is of an inverse Fourier transform variety, will be further simplified by use of convolution

$$\int_{-\infty}^{\infty} u(Z-\zeta)\, v(\zeta)\, d\zeta = \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\xi)\, V(\xi) e^{-jZ\xi}\, d\xi \quad , \tag{39}$$

with

$$\begin{aligned} u(\zeta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(\xi) e^{-j\zeta\xi}\, d\xi \\ v(\zeta) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\xi) e^{-j\zeta\xi}\, d\xi \quad . \end{aligned} \tag{40}$$

Therefore, from the integral in (38) one has

$$u(Z-\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-j\zeta(Z-\zeta)}}{\zeta^2+\gamma_1^2+a^2}\, d\xi \tag{41}$$

$$v(\zeta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-j\xi\zeta}}{\sqrt{\xi^2+\gamma_1^2}}\, d\xi \tag{42}$$

To evaluate (41) for $Z>\zeta$, one closes the contour at infinity in the lower half plane and takes out the residue at ξ_{p1} . For $Z<\zeta$ the contour is deformed in the upper half plane and the pole ξ_{p2} is captured with the overall uniformly valid result

$$u(Z-\zeta) = \frac{e^{-\sqrt{\gamma_1^2+a^2}\, |Z-\zeta|}}{2\sqrt{\gamma_1^2+a^2}} \quad , \quad \text{Re}\sqrt{\gamma_1^2+a^2} > 0 \; . \tag{43}$$

Evaluation of $v(\zeta)$ yields

$$v(\zeta) = \frac{1}{\pi} K_o(\gamma_1|\xi|) \; , \tag{44}$$

where K_o is the modified Bessel function of the second kind and zeroth order.

Consequently one has,

$$G(Z) = \frac{-j\pi}{a} \quad \frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} - \int_{-\infty}^{\infty} \frac{e^{-\sqrt{\gamma_1^2+a^2}\,|Z-\zeta|}}{\sqrt{\gamma_1^2+a^2}} \, K_o(\gamma_1|\zeta|)d\zeta, \qquad \begin{aligned} &\mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0 \\ &\mathrm{Re}\,\gamma_1 > 0 \\ &\mathrm{Im}\,\gamma_1 < 0 \end{aligned} \tag{45}$$

To further simplify the integral I(Z) in the (44) , one has

$$I(Z) = \frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} \int_{-\infty}^{Z} d\zeta \; e^{+\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1|\zeta|)d\zeta + \frac{e^{\sqrt{\gamma_1+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} \int_{Z}^{\infty} e^{-\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1|\zeta|)d\zeta \tag{46}$$

or

$$I(Z) = \frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} \left[\int_0^{\infty} e^{-\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)\,d\zeta + \int_0^{Z} e^{\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)d\zeta \right] +$$

$$+ \frac{e^{\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} \left[\int_0^{\infty} e^{-\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)d\zeta - \int_0^{Z} e^{-\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)d\zeta \right] =$$

$$= \frac{2\cosh\sqrt{\gamma_1^2+a^2}\,Z}{\sqrt{\gamma_1^2+a^2}} \int_0^{\infty} e^{-\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)d\zeta +$$

$$+ \frac{e^{-\sqrt{\gamma_1+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} \int_0^{Z} e^{\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)d\zeta - \frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} \int_0^{Z} e^{-\sqrt{\gamma_1^2+a^2}\,\zeta} \, K_o(\gamma_1\zeta)d\zeta \tag{47}$$

But from [4]

$$\int_0^\infty d\zeta\, e^{-\sqrt{\gamma_1^2+a^2}\,\zeta}\, K_o(\gamma_1\zeta) = \frac{\ln\left(\dfrac{a+\sqrt{\gamma_1^2+a^2}}{\gamma_1}\right)}{a} \qquad (48)$$

$$\mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0 > -\,\mathrm{Re}\,\gamma_1 \quad ,$$

where the principal value of $\ln$ is understood, $\mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0$, and the convergence of the integral is assured since both $\mathrm{Re}\sqrt{\gamma_1^2+a^2} > 0$ and $\mathrm{Re}\,\gamma_1 > 0$.

Thus, the evaluation of G(Z) is now reduced to the evaluation of the two finite integrals

$$I_\pm = \int_0^Z d\zeta\, e^{\pm\sqrt{\gamma_1^2+a^2}\,\zeta}\, K_o(\gamma_1\zeta)\, d\zeta \quad . \qquad (49)$$

For small values of $\gamma_1 Z$ i.e., $|\gamma_1 Z| << 1$, $K_o(w)$ may be approximated by $-\ln\gamma w/2$ where γ is the Euler constant. This leads to expressions involving exponential integrals. Otherwise, in the dispersion relation, which in view of (26), (27) and (C1), contains $I_\pm$ and their derivatives with respect to Z , the most rapidly convergent procedure valid for any value $\gamma_1 Z$ is to evaluate $I_\pm$ numerically with $\gamma_1 = \sqrt{\beta^2-k^2}$ as a β dependent variable, and $\sqrt{\gamma_1^2+a^2} = \dfrac{-jk}{\sqrt{1+\epsilon}}$ as a β independent parameter.

Altogether,

$$G(Z) = j\left\{\frac{-j\pi}{a}\,\frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}} - \frac{2\cosh(\sqrt{\gamma_1^2+a^2}\,Z)}{\sqrt{\gamma_1^2+a^2}}\,\frac{\ln\left(\dfrac{a+\sqrt{\gamma_1^2+a^2}}{\gamma_1}\right)}{a}\right.$$

$$\left. - \frac{e^{-\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}}\int_0^Z e^{\sqrt{\gamma_1^2+a^2}\,\zeta}\, K_o(\gamma_1\zeta)\,d\zeta + \frac{e^{\sqrt{\gamma_1^2+a^2}\,Z}}{\sqrt{\gamma_1^2+a^2}}\int_0^Z e^{-\sqrt{\gamma_1^2+a^2}\,\zeta}\, K_o(\gamma_1\zeta)\,d\zeta\right\} . \qquad (50)$$

With G(Z) given by (49), H(Z) is calculated in a similar fashion as F(Z), by expanding κ_2 via (21).

Conclusions

A rigorous formalism has been presented, that permits setting up of the dispersion relation for a periodically slotted coaxial cable above, and parallel to, a lossy ground in the surface wave region. A simple modification of the procedure will allow to account for one or more fast space harmonics and thus to obtain expressions valid for periodic leaky cables. The expressions may be easily analytically continued, (in contrast to those in Ref. [2], in the case when the Sommerfeld poles as a function of complex propagation constant β would be required to cross the real η axis. An interesting observation is appropriate here. Consider expression (28), in the complex η plane with its Sommerfeld poles at $\eta_p = \pm a$ in the Im a < o which are a function of β . As recognized in Ref. [2] , the poles move in the complex η plane as a function of the complex axial propagation constant β, which is governed by the dispersion relation of the periodically slotted structure. If the pole at $\eta_p = a$, as a function of β, crosses the real axis into the upper half plane, the pole at $\eta_p = -a$ crosses the real η axis simultaneously into the lower half plane and the representation (28) is a discontinuous function of $\beta(k)$. Such functions however are not admissible in the dispersion relation and therefore the function G when $\eta_p = a(\beta)$ approaches the real axis from below, must be analytically continued, as a function of β . A similar situation occurs in Landau damping [5], and the procedure is analogous here. As the poles $\pm a$ cross the $\eta_i = 0$ axis the contour is indented as shown in Fig. 5a. After the pole $\eta_p = a$ crossed into the upper half plane the appropriate contour is shown in Fig. 5b.

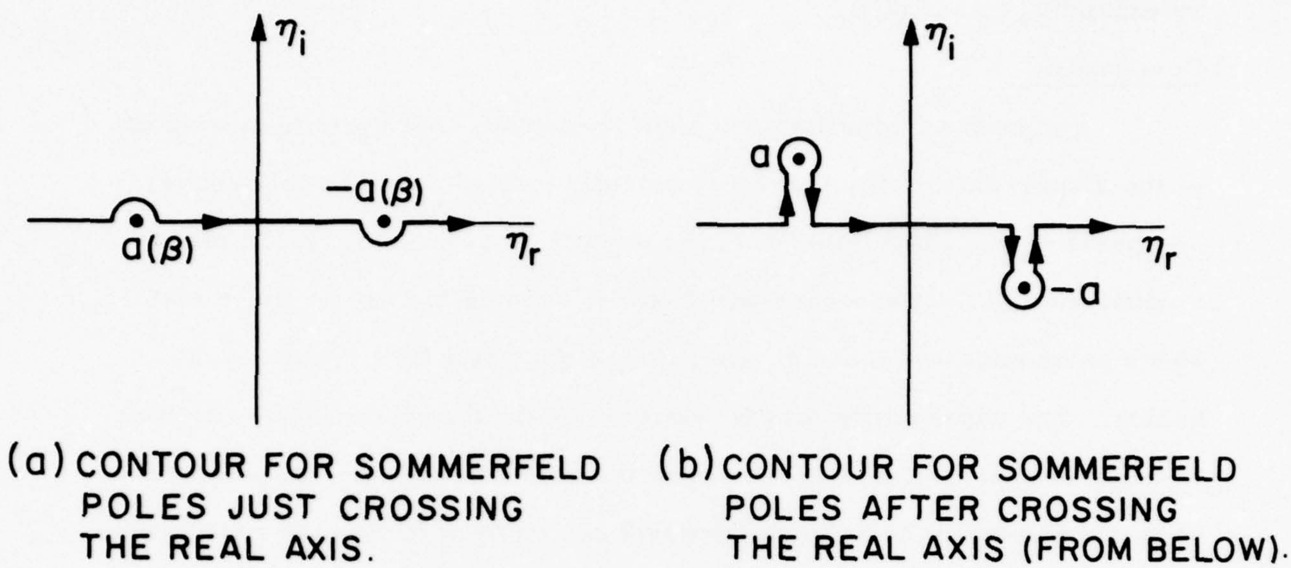

Fig. 5. Analytic Continuation of $G(y, Z, \beta)$.

The deformation of contour provides a simple means of analytic continuation of $G(y, Z, \beta)$ whereby the appropriate pole $\eta_p = a$ is tracked even if it crosses into the upper half η plane. Physically, after the pole $\eta_p = a$ crossed the real η axis, so that now $\mathrm{Im}\,\eta_p > 0$, such a pole represents a leaky wave in the y direction. This is obvious, when in (26) the contour would be deformed into the lower branch cut, prior to Taylor series expansion of the $e^{-j\eta y}$.

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Appendix A

Summary of Radial Line Formalism for a Unit Cell in Cylindrical Geometry (based on Ref. [3]).

When a scalar and a vector multiplication with a radial unit vector $\underline{\rho}_o$ is applied to the Maxwell's equations

$$\nabla \times \underline{H} = j\omega\epsilon\, \underline{E}$$
$$\nabla \times \underline{E} = -j\omega\mu_o \underline{H} - \underline{M}, \qquad A(1)$$

the following set of coupled equations arises for the transverse to ρ field components $\underline{E}_t$ and $\underline{H}_t$.

$$-\nabla_\rho \underline{E}_t = j\omega\mu \left[\underline{\underline{1}} + \frac{\nabla_t \nabla_t}{k^2}\right] \cdot (\underline{H}_t \times \underline{\rho}_o) + \underline{M}_t \times \underline{\rho}_o$$

$$-\nabla_\rho \underline{H}_t = j\omega\epsilon \left[\underline{\underline{1}} + \frac{\nabla_t \nabla_t}{k^2}\right] \cdot (\underline{\rho}_o \times \underline{E}_t) , \qquad A(2)$$

where $\nabla_t = \underline{\Phi}_o \dfrac{1}{\rho} \dfrac{\partial}{\partial \varphi} + \underline{z}_o \dfrac{\partial}{\partial z}$.

The unit cell orthonormal vector mode functions that scalarize this set of equations are

$$\left.\begin{aligned} \underline{e}'_{mn}(\underline{r}) &= \frac{1}{\sqrt{2\pi}\, d}\left[-\frac{\beta_n m}{\kappa_n \rho}\underline{\Phi}_o + \underline{z}_o\right] e^{-jm\Phi}\, e^{-j\beta_n z} \\ \underline{h}'_{mn}(\underline{r}) &= \frac{1}{\sqrt{2\pi}\, d}\left[-\frac{1}{\rho}\underline{\Phi}_o\right] e^{-jm\Phi}\, e^{-j\beta_n z} \end{aligned}\right\} \begin{array}{l}\text{E-type modes} \\ (H_z = 0)\end{array} \qquad A(3)$$

$$\left.\begin{aligned} \underline{e}''_{mn}(\underline{r}) &= \frac{1}{\sqrt{2\pi d}}\left[\frac{1}{\rho}\underline{\Phi}_o\right] e^{-jm\Phi}\, e^{-j\beta_n z} \\ \underline{h}''_{mn}(\underline{r}) &= \frac{1}{\sqrt{2\pi}\, d}\left[-\frac{\beta_n m}{\kappa_n^2 \rho}\underline{\Phi}_o + \underline{z}_o\right] e^{-jm\Phi}\, e^{-j\beta_n z} \end{aligned}\right\} \begin{array}{l}\text{H-type modes} \\ (E_z = 0)\end{array} ,$$

their orthonormality condition being

$$\int_{\substack{\text{unit cell}\\ \text{cross section}}} \underline{e}_i \times \underline{h}_j^* \cdot \underline{\rho}_o \, dS = \delta_{ij} \; . \qquad A(4)$$

In (A3), $\varkappa_n^2 = k^2 - \beta_n^2$, $\underline{\Phi}_o$ and $\underline{z}_o$ are unit vectors along the local Φ and the axial direction and d is the period. Also

$$\nabla_\rho \underline{A}_t = \underline{\Phi}_o \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \, A_\Phi) \right] + \underline{z}_o \frac{\partial A_z}{\partial z} \; . \qquad A(5)$$

Substitution of $\underline{E}_{tmn}(\underline{r}) = V_{mn}(\rho) \, \underline{e}_{mn}(\rho, \Phi, z)$, $\underline{\dot{H}}_{tmn}(\underline{r}) = \dot{I}_{mn}(\rho) \, \underline{h}_{mn}(\rho, \Phi, z)$ into (A2) yields the following radial transmission line equations for the modal voltages and currents (when $\underline{M}$ has no radial component)

$$- \frac{d}{d\rho} V_{mn}(\rho) = j\varkappa_{mn}(\rho) \, Z_{mn}(\rho) \, I_{mn}(\rho) + v_{mn} \qquad A(6)$$

$$- \frac{dI_{mn}(\rho)}{d\rho} = j\varkappa_{mn}(\rho) \, Y_{mn}(\rho) \, V_{mn}(\rho) \quad ,$$

where
$$Z'_{mn}(\rho) = \frac{1}{Y'_{mn}(\rho)} = \frac{\varkappa_n}{\omega\epsilon_\rho \, \varkappa_{mn}(\rho)} \qquad A(7)$$

$$Y''_{mn}(\rho) = \frac{1}{Z''_{mn}(\rho)} = \frac{\varkappa_n^2}{\omega\mu_\rho \, \varkappa_{mn}(\rho)} \qquad A(8)$$

and

$$\varkappa''_{mn}(\rho) = \varkappa''_{mn}(\rho) = \varkappa^2_{mn}(\rho) = \varkappa_n^2 - \frac{m^2}{\rho^2} \; . \qquad A(9)$$

From (A 6) one finds that for $M = 0$ $V'_{mn}(\rho)$ and $I''_{mn}(\rho)$ satisfy the Bessel differential equation

$$\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \varkappa_n^2 - \frac{m^2}{\rho^2}\right]\begin{pmatrix} V'_{mn}(\rho) \\ I''_{mn}(\rho)\end{pmatrix} = 0 \quad , \tag{A(10)}$$

while the generator voltages v'_{mn} and v''_{mn} are obtained for the case of transverse magnetic current sources via an expansion $\underline{M} \times \underline{\rho}_o = \Sigma(v'_{mn}\underline{e}'_{mn} + v''_{mn}\underline{e}''_{mn})$ so that using (A4)

$$v'_{mn}(\rho) = \iint_{\substack{\text{unit cell} \\ \text{cross-section}}} \underline{M}_t(\underline{r}') \cdot \underline{h}^*_{mn}(\rho)\, ds \tag{A(11)}$$

A tangential magnetic point source $\underline{M}_t(\underline{r}')$ on an exterior of an unperforated coaxial surface in a unit cell may be represented by

$$\underline{M}_t(\underline{r}') = \underline{u}_o\, M\, \frac{\delta(\rho - a)\,\delta(\varphi - \varphi')\,\delta(z - z')}{a} \tag{A(12)}$$

where $\underline{u}_o$ is a unit vector in the direction of $\underline{M}_t$.

Hence,

$$\begin{aligned} v'_{mn}(\rho) &= M\, \underline{u}_o \cdot \underline{h}'^*_{mn}(\underline{r}')\, \delta(\rho - a) = v'_{mn}\, \delta(\rho - a) \\ v''_{mn}(\rho) &= M\, \underline{u}_o \cdot \underline{h}''^*_{mn}(\underline{r}')\, \delta(\rho - a) = v''_{mn}\, \delta(\rho - a) \end{aligned} \tag{A(13)}$$

Upon integration of equations(A6) one finds that

$$\begin{aligned} -(V'_{mn}(a^+) - V'_{mn}(a)) &= v'_{mn} \\ -(V''_{mn}(a^+) - V''_{mn}(a)) &= v''_{mn} \end{aligned} \tag{A(14)}$$

and since $V'_{mn}(a) = V''_{mn}(a) = 0$, in view of the perfectly conducting cylindrical surface at $\rho = a$,

$$V'_{mn}(a^+) = -v'_{mn}$$

$$V''_{mn}(a^+) = -v''_{mn} \qquad \text{A(15)}$$

In view of (A10), for the exterior of a cable in free space

$$V'_{mn}(\rho) = A\, H_m^{(2)}(\kappa_n \rho) \quad . \qquad \text{A(16)}$$

Hence,

$$V'_{mn}(a^+) = A\, H_m^{(2)}(\kappa_n a) \qquad \text{A(17)}$$

and therefore,

$$V'_{mn}(\rho) = -\frac{M\, \underline{u}_o \cdot \underline{h}'^{*}_{mn}(\underline{r}')}{H_m^{(2)}(\kappa_n a)} H_m^{(2)}(\kappa_n \rho) \quad . \qquad \text{A(18)}$$

Therefore,from ( )

$$I'_{mn}(\rho) = \frac{-M\, \underline{u}_o \cdot \underline{h}^{*}_{mn}(\underline{r}')}{H_m^{(2)}(\kappa_n a)} \cdot j \frac{H_m^{(2)'}(\kappa_n \rho)\, \omega\varepsilon\rho}{\kappa_n} = \frac{-j\omega\varepsilon\rho M}{\kappa_n} \frac{H_m^{(2)'}(\kappa_n \rho)}{H_m^{(2)}(\kappa_n a)} \underline{u}_o \cdot \underline{h}'^{*}_{mn}(\underline{r}) \quad . \qquad \text{A(19)}$$

To calculate $I''_{mn}(\rho)$ one has from (A10)

$$I''_{mn}(\rho) = B\, H_m^{(2)}(\kappa_n \rho) \quad , \qquad \text{A(20)}$$

so that

$$I''_{mn}(a^+) = B\, H_m^{(2)}(\kappa_n a) \qquad \text{A(21)}$$

and therefore

$$I''_{mn}(\rho) = I''_{mn}(a^+) \; \frac{H_o^{(2)}(\varkappa_n \rho)}{H_o^{(2)}(\varkappa_n a)} \quad . \qquad \text{A(22)}$$

From (A6) one has via (A 15)

$$V''_{mn}(\rho) = j \frac{1}{\varkappa''_{mn}(\rho)\, Y''_{mn}(\rho)} \frac{d}{d\rho} I''_{mn}(\rho) = \frac{j\omega\mu\rho}{\varkappa_n} I''_{mn}(a^+) \frac{H_m^{(2)'}(\varkappa_n \rho)}{H_m^{(2)}(\varkappa_n a)}$$

or

$$V''_{mn}(a^+) = \frac{j\omega\mu a}{\varkappa_n} I''_{mn}(a^+) \frac{H_m^{(2)'}(\varkappa_n a)}{H_m^{(2)}(\varkappa_n a)} = - M \underline{u}_o \cdot \underline{h}''^{*}(\underline{r}') \qquad \text{A(23)}$$

Hence,

$$I''_{mn}(a^+) = - \frac{M u_o \cdot \underline{h}''^{*}_{mn}(\underline{r}')}{j\omega\mu a} \varkappa_n \frac{H_m^{(2)}(\varkappa_n a)}{H_m^{(2)'}(\varkappa_n a)} \qquad \text{A(24)}$$

and

$$I''_{mn}(\rho) = - \frac{\underline{M}_t \cdot \underline{h}''^{*}_{mn}(\underline{r}')}{j\omega\mu a} \varkappa_n \frac{H_m^{(2)}(\varkappa_n \rho)}{H_m^{(2)'}(\varkappa_n a)} \quad . \qquad \text{A(25)}$$

Consequently,

$$\underline{H}_t(\underline{r}, \underline{r}') = \underline{\underline{G}}_{EF}(\underline{r}, \underline{r}') \cdot \underline{M}_t = - \left\{ \sum_{mn=-\infty}^{\infty} Y'_n(\rho) \frac{H_m^{(2)'}(\varkappa_n \rho)}{H_m^{(2)}(\varkappa_n a)} \underline{h}'_{mn}(\underline{r}) \underline{h}'^{*}_{mn}(\underline{r}') \right.$$

$$\left. + \sum_{mn=-\infty}^{\infty} Y''_n(a) \frac{H_m^{(2)}(\varkappa_n \rho)}{H_m^{(2)'}(\varkappa_n a)} \underline{h}''_{mn}(\underline{r})\, \underline{h}''^{*}_{mn}(\underline{r}') \right\} \cdot \underline{M}_t \qquad \text{A(26)}$$

Hence, the transverse part of the exterior dyadic Green's Function for a coaxial cable in free space is given by

$$\underline{\underline{G}}_{EF}(\underline{r},\underline{r}';\beta) = \left\{ \sum_{mn=-\infty}^{\infty} Y_n'(\rho) \frac{H_m^{(2)'}(\varkappa_n\rho)}{H_m^{(2)}(\varkappa_n a)} \underline{h}'_{mn}(r)\, \underline{h}'^{*}_{mn}(r') \; + \right.$$

$$\left. \sum_{mn=-\infty}^{\infty} Y_n''(a) \frac{H_m^{(2)}(\varkappa_n\rho)}{H_m^{(2)'}(\varkappa_n a)} \underline{h}''_{mn}(\underline{r})\, \underline{h}''^{*}_{mn}(\underline{r}') \right\} \qquad A(27)$$

with

$$Y_n' = \frac{j\omega\varepsilon\rho}{\varkappa_n}$$

$$Y_n'' = \frac{\varkappa_n}{j\omega\mu\rho} \quad . \qquad A(28)$$

It can be readily verified that $\underline{\underline{G}}_{EF}$ can be derived from two scalar potentials π' and π''

$$\pi' = \frac{j\omega\varepsilon}{\sqrt{2\pi d}} \sum_{mn=-\infty}^{\infty} \frac{1}{\varkappa_n^2} \frac{H_m^{(2)}(\varkappa_n\rho)}{H_m^{(2)}(\varkappa_n a)} e^{-jm\Phi} e^{-j\beta_n z} \qquad A(29)$$

$$\pi'' = -\frac{1}{j\omega\mu\rho} \frac{1}{\sqrt{2\pi d}} \sum_{mn=-\infty}^{\infty} \frac{1}{\varkappa_n} \frac{H_m^{(2)}(\varkappa_n\rho)}{H_m^{(2)'}(\varkappa_n a)} e^{-jm\Phi} e^{-j\beta_n z} \quad . \qquad A(30)$$

Thus, both π' and π'' consist of terms of the form

$$C_{mn} H_m^{(2)}(\varkappa_n\rho)\, e^{-jm\Phi} e^{-j\beta_n z} \quad , \qquad A(31)$$

each representing a Hertz potential of an electric (π'_{mn}) or a magnetic (π''_{mn}) phased line source. Expression for the interior dyadic Green's Function $G_I(\underline{r},\underline{r}';\beta)$ may be derived in a similar manner.

Appendix B

Hertz Potentials for Phased Line Sources Above a Dielectric Half Space.

The potentials for an electric line source appear in [2] and were originally obtained by J. R. Wait. For completeness we rederive them for the electric line source, and, in addition obtain expression for a magnetic line source.

1. Electric Current Source at $\underline{\rho}' = (y', z')$ above the interface.

We assume that $\underline{E}$ and $\underline{H}$ are derived from two Hertz potentials $\Pi'\underline{x}_o$ and $\Pi''\underline{x}_o$ such that

In the air region	In the dielectric
$\underline{H}'^{+} = j\omega\epsilon_o \nabla \times \Pi'^{+}\underline{x}_o$	$H'^{-} = j\omega\epsilon\epsilon_o \nabla \times \Pi'^{-} x_o$
$\underline{E}'^{+} = \nabla \times \nabla \times \Pi'^{+}\underline{x}_o$	$\underline{E}'^{-} = \nabla \times \nabla \times \Pi'^{-}\underline{x}_o$
$\underline{E}''^{+} = -j\omega\mu_o \nabla \times \Pi''^{+}\underline{x}_o$	$\underline{E}''^{-} = -j\omega\mu_o \nabla \times \Pi''^{-}\underline{x}_o$
$\underline{H}''^{+} = \nabla \times \nabla \times \Pi''^{+} x_o$	$\underline{H}''^{-} = \nabla \times \nabla \times \Pi''^{-}\underline{x}_o$
$\nabla^2\Pi'^{+} + k^2\Pi'^{+} = -\dfrac{J}{j\omega\epsilon_o}$	$\nabla^2\Pi'^{-} + k^2\epsilon\Pi'^{-} = 0$
$\nabla^2\Pi''^{+} + k^2\Pi''^{+} = 0$	$\nabla^2\Pi''^{-} + k^2\epsilon\Pi''^{-} = 0$

B(1)

The primary electric line source fields are generated by Π'_{inc}. The scattered fields in the air space are derived from $\Pi_s'^{+}$ and $\Pi_s''^{+}$ such that $\Pi'^{+} = \Pi'_{inc} + \Pi_s'^{+}$, $\Pi''^{+} = \Pi''_s$, with $\nabla^2\Pi'_{inc} + k^2\Pi'_{inc} = \dfrac{-J}{j\omega\epsilon_o}$ and $\nabla^2\Pi_s'^{+} + k^2\Pi_s'^{+} = 0$ as well as $\nabla^2\Pi_s''^{+} + k^2\Pi_s''^{+} = 0$.

In the dielectric half space one has $\Pi'^{-} = \Pi_s^{-}$, $\Pi''^{-} = \Pi_s''^{-}$, both of which satisfy the homogenous Helmholtz equation.

If $\underline{J} = J_o e^{-j\beta x}\delta(y-y')\delta(z-z')\underline{x}_o$ B(2)

then

$$\Pi'_{inc} = \frac{I_{oe}^{-j\beta x}}{j\omega\varepsilon_o} \cdot \frac{-j}{4} H_o^{(2)}\left(\sqrt{k^2-\beta^2}\sqrt{(y-y')^2+(z-z')^2}\right) \qquad \text{B(3)}$$

$$\frac{-I_{oe}^{-j\beta x}}{\omega\varepsilon_o 4\Pi} \int_{\infty}^{\infty} \frac{e^{-j\eta y} e^{-j\varkappa_1 |z-z'|}}{\varkappa_1} d\eta \ ; \quad \varkappa_1 = \sqrt{k^2-\beta^2-\eta^2}$$

$$\underline{\mathrm{Im}\varkappa < 0} \ .$$

Setting, in accordance with the radiation condition,

$$\Pi'_{inc} = e^{-j\beta x} \int_{-\infty}^{\infty} A(\eta) e^{-j\eta y} e^{+j\varkappa_1(z-z')} d\eta \ ; \quad 0 < z < z'$$

$$\Pi_s'^{+} = e^{-j\beta x} \int_{-\infty}^{\infty} \left(B(\eta) - A(\eta)\right) e^{-j\eta y} e^{-j\varkappa_1(z+z')} d\eta$$

$$\Pi_s''^{+} = e^{-j\beta x} \int_{-\infty}^{\infty} C(\eta) e^{-j\eta y} e^{-j\varkappa_1(z+z')} d\eta \qquad \text{B(4)}$$

$$\Pi_s'^{-} = e^{-j\beta x} \int_{-\infty}^{\infty} D(\eta) e^{-j\eta y} e^{-j\varkappa_2 z} d\eta$$

$$\Pi_s''^{-} = e^{-j\beta x} \int_{-\infty}^{\infty} E(\eta) e^{-j\eta y} e^{-j\varkappa_2 z} d\eta$$

and imposing continuity of E_x, E_y, H_x and H_y at $z = 0$ one obtains, in view of (B1) and (B4) the following set of linear inhomogenous equations for the determination of the transforms B, C, D and E:

$$(k^2\varepsilon - \beta^2)D = (k^2 - \beta^2)Be^{-j\varkappa_1 z'} \qquad \text{B(5)}$$

$$(k^2\varepsilon - \beta^2)E = (k^2 - \beta^2)Ce^{-j\varkappa_1 z'} \qquad \text{B(6)}$$

$$-\beta\eta B e^{-j\varkappa_1 z'} - \omega\mu_o \varkappa_1 C e^{-j\varkappa_1 z'} = -\beta\eta D + \omega\mu_o \varkappa_2 E \tag{B(7)}$$

$$-2\omega\varepsilon_o \varkappa_1 e^{-j\varkappa_1 z'} A + \omega\varepsilon_o \varkappa_1 e^{-j\varkappa_1 z'} B - \beta\eta e^{-j\varkappa_1 z'} C = -\omega\varepsilon\varepsilon_o \varkappa_2 D - \beta\eta E \; . \tag{B(8)}$$

From (B3) one has

$$A(\eta) = \frac{-I_o}{4\pi\omega\varepsilon_o \varkappa_1} \quad . \tag{B(9)}$$

Elimination of D and E yields

$$-2\omega\varepsilon_o\left(\varepsilon - \left(\frac{\beta}{k}\right)^2\right)A = -\omega\varepsilon_o\left[\left(\varepsilon - \left(\frac{\beta}{k}\right)^2\right)\varkappa_1 + \varepsilon\varkappa_2\left(1 - \left(\frac{\beta}{k}\right)^2\right)\right]B + \beta\eta(\varepsilon - 1)C$$

$$0 = \beta\eta(\varepsilon -)B + \omega\mu_o\left[\left(\varepsilon - \left(\frac{\beta}{k}\right)^2\right)\varkappa_1 + \varkappa_2\left(1 - \left(\frac{\beta}{k}\right)^2\right)\right]C \; . \tag{B(10)}$$

After certain amount of manipulations, the determinant of the system Δ may be shown to reduce to

$$\Delta = -\left(\varepsilon - \left(\frac{\beta}{k}\right)^2\right)\left(1 - \left(\frac{\beta}{k}\right)^2\right)\left(\varkappa_1 + \varkappa_2\right)\left(\varkappa_2 + \varepsilon\varkappa_1\right) \quad . \tag{B(11)}$$

Consequently, from (B9), (B10) and (B11) one has

$$B = \frac{-2I_o}{4\pi\omega\varepsilon_o}\left[\frac{1}{\varkappa_1 + \varkappa_2} - \frac{\left(\frac{\beta}{k}\right)^2}{\varkappa_2 + \varepsilon\varkappa_1}\right]\frac{1}{1 - \left(\frac{\beta}{k}\right)^2} \tag{B(12)}$$

$$C = \frac{I_o \beta/k}{2\pi k\left(1 - \left(\frac{\beta}{k}\right)^2\right)}\left[\frac{1}{\varkappa_2 + \varkappa_1} - \frac{1}{\varkappa_2 + \varepsilon\varkappa_1}\right]\frac{\eta}{\varkappa_1} \tag{B(13)}$$

Thus,

$$\Pi'^{+} = \frac{-I_o}{4\omega\epsilon_o} e^{-j\beta x} \left\{ H_o^{(2)} \sqrt{k^2-\beta^2} \sqrt{(y-y')^2+(z-z')^2} - H_o^{(2)}\left(\sqrt{k^2-\beta^2}\sqrt{(y-y')^2+(z+z')^2}\right) + \right.$$

$$\left. + \frac{2}{\pi\left(1-\left(\frac{\beta}{k}\right)^2\right)} \int_{-\infty}^{\infty} \left[\frac{1}{\varkappa_1+\varkappa_2} - \frac{1}{\varkappa_2+\epsilon\varkappa_1}\right] e^{-j\eta y} e^{-j\varkappa_1(z+z')} d\eta \right\} \quad . \qquad B(14)$$

The expressions differ by a factor of 2 in front of the $\{\ \}$ bracket and by a factor of $\frac{1}{-j} \to \frac{1}{i}$ in front of the integral from that in Eq. (1.2) of Ref. [2].

$$\Pi''^{+} = \frac{I_o\beta/k\, e^{-j\beta x}}{2\pi k\left(1-\left(\frac{\beta}{k}\right)^2\right)} \int_{-\infty}^{\infty} \left[\frac{1}{\varkappa_2+\varkappa_1} - \frac{1}{\varkappa_2+\epsilon\varkappa_1}\right]\frac{\eta}{\varkappa_1} e^{-j\eta y} e^{j\varkappa_1(z+z')} d\eta \quad . \qquad B(15)$$

This expression checks with (1.4) of Ref. [2].

2. Magnetic Current Line Source.

$$\underline{J}_m = M\, e^{-j\beta x}\, \delta(y-y')\, \delta(z-z')\, \underline{x}_o \qquad B(16)$$

$\underline{E}$ and $\underline{H}$ due to (B16) in free space is derived from $\Pi''_{inc}\underline{x}_o$,

where

$$\nabla^2 \hat{\Pi}''_{inc} + k^2 \hat{\Pi}''_{inc} = \frac{J_m}{j\omega\mu} \quad , \qquad B(17)$$

with the radiation condition at infinity.

Thus,

$$\hat{\Pi}''_{inc} = \frac{Me^{-j\beta x}}{j\omega\mu_o}\left(\frac{j}{4} H_o^{2}{}'\left(\sqrt{k^2-\beta^2}\sqrt{(y-y')^2+(z-z')^2}\right)\right) =$$

$$= \frac{Me^{-j\beta x}}{4\pi\omega\mu_o}\int_{-\infty}^{\infty} \frac{e^{-j\eta y}\, e^{-j\varkappa_1 |z-z'|}}{\varkappa_1}\, d\eta \quad . \qquad B(18)$$

A magnetic phased line source parallel to an air dielectric interface excites fields that are derivable, similarly to the case of an electric line source, from two Hertz potentials via the first four relations in B(1), the difference being that now in addition to (B17)

$$\hat{\Pi}''^{+} = \hat{\Pi}''_{inc} + \hat{\Pi}''^{+}_{s}$$

$$\hat{\Pi}'^{+} = \hat{\Pi}'^{+}_{s} \qquad B(19)$$

The relations analogous to (B4) are

$$\hat{\Pi}''_{inc} = e^{-j\beta x}\int_{-\infty}^{\infty} \hat{A}(\eta) e^{-j\eta y} e^{j\varkappa_1 (z+z')}\, d\eta \quad ; \quad 0 < z < z'$$

$$\hat{\Pi}''^{+}_{s} = e^{-j\beta x}\int_{-\infty}^{\infty} (\hat{B}(\eta) - \hat{A}) e^{-j\eta y} e^{-j\varkappa_1 (z+z')}\, d\eta$$

$$\hat{\Pi}'^{+}_{s} = e^{-j\beta x}\int_{-\infty}^{\infty} \hat{C}(\eta) e^{-j\eta y} e^{-j\varkappa_1 (z+z')}\, d\eta \qquad B(20)$$

$$\hat{\Pi}''^{-}_{s} = e^{-j\beta x}\int_{-\infty}^{\infty} \hat{D}(\eta) e^{-j\eta y} e^{-j\varkappa_2 z}\, d\eta$$

$$\hat{\Pi}'^{-}_{s} = e^{-j\beta x}\int_{-\infty}^{\infty} \tilde{E}(\eta) e^{-j\eta y} e^{j\varkappa_2 z}\, d\eta$$

Application of continuity of the transverse to z fields at $z = 0$ yields via (B1) the following equations:

$$\left.\begin{aligned} \left[1 - \left(\frac{\beta}{k}\right)^2\right] \hat{B} e^{-j\varkappa_1 z'} &= \left(\epsilon - \left(\frac{\beta}{k}\right)^2\right) \hat{D} \\ \left[1 - \left(\frac{\beta}{k}\right)^2\right] \hat{C} e^{-j\varkappa_1 z'} &= \left(\epsilon - \left(\frac{\beta}{k}\right)^2\right) \hat{E} \end{aligned}\right\} \qquad B(21)$$

$$\left.\begin{aligned} &\omega\epsilon_o \varkappa_1 e^{-j\varkappa_1 z'} \hat{C} - \beta\eta e^{-j\varkappa_1 z'} \hat{B} = - \omega\epsilon_o \epsilon \varkappa_2 \hat{E} - \beta\eta \hat{D} \\ &-\beta\eta e^{-j\varkappa_1 z'} \hat{C} - \omega\mu_o \varkappa_1 e^{-j\varkappa_1 z'} \hat{B} + 2\omega\mu_o \varkappa_1 e^{-j\varkappa_1 z'} \hat{A} = -\beta\eta\hat{E} + \omega\mu_o \varkappa_2 \hat{D} \end{aligned}\right\} \qquad B(22)$$

Elimination of $\hat{D}$ and $\hat{E}$ between (B21) and (B22) yields

$$\begin{aligned} + 2\omega\mu_o \varkappa_1 \hat{A} &= \omega\mu_o\left[\varkappa_1\left(\epsilon - \left(\frac{\beta}{k}\right)^2\right) + \varkappa_2\left(1 - \left(\frac{\beta}{k}\right)^2\right)\right] \hat{B} + \beta\eta(\epsilon - 1)\hat{C} \\ 0 &= - \beta\eta(\epsilon - 1)\hat{B} + \omega\epsilon_o\left[\varkappa_1\left(\epsilon - \left(\frac{\beta}{k}\right)^2\right) + \epsilon\varkappa_2\left(1 - \left(\frac{\beta}{k}\right)^2\right)\right] \hat{C} \end{aligned} \qquad B(23)$$

Similarly to (B11) the system determinant $\hat{\Delta}$ is

$$\hat{\Delta} = + k^2\left(\epsilon - \left(\frac{\beta}{k}\right)^2\right)\left(\varkappa_1 + \varkappa_2\right)\left(\varkappa_2 + \epsilon\varkappa_1\right) \qquad B(24)$$

and

$$\begin{aligned} \hat{B} &= 2\varkappa_1 \hat{A} \frac{\varkappa_1\left(\epsilon - \left(\frac{\beta}{k}\right)^2\right) + \epsilon\varkappa_2\left(1 - \left(\frac{\beta}{k}\right)^2\right)}{\left(\epsilon - \left(\frac{\beta}{k}\right)^2\right)\left(1 - \left(\frac{\beta}{k}\right)^2\right)\left(\varkappa_1 + \varkappa_2\right)\left(\varkappa_2 + \epsilon\varkappa_1\right)} = \\ &= \frac{M}{2\pi\omega\mu_o\left(\epsilon - \left(\frac{\beta}{k}\right)^2\right)\left(1 - \left(\frac{\beta}{k}\right)^2\right)} \left[\frac{\epsilon\left(1 - \left(\frac{\beta}{k}\right)^2\right) - \left(\frac{\beta}{k}\right)^2}{\varkappa_2 + \epsilon\varkappa_1} + \frac{\left(\frac{\beta}{k}\right)^2}{\varkappa_2 + \varkappa_1}\right], \end{aligned} \qquad B(25)$$

while

$$\hat{C} = \frac{M\frac{\beta}{K}(\epsilon - 1)\ \eta}{2\pi k\left(\epsilon - \left(\frac{\beta}{k}\right)^2\left(1 - \left(\frac{\beta}{k}\right)^2\right)\left(\varkappa_1 + \varkappa_2\right)\left(\varkappa_2 + \epsilon\varkappa_1\right)\right)} \qquad \text{B(26)}$$

Altogether,

$$\Pi''^{+} = \frac{Me^{-j\beta x}}{4\omega\mu_o}\left[H_o^2\left(\sqrt{k^2-\beta^2}\sqrt{(y-y')^2+(z-z')^2}\right)-H_o^2\left(\sqrt{k^2-\beta^2}\sqrt{(y-y')^2+(z+z')^2}\right)\right.+ \qquad \text{B(27)}$$

$$+\frac{Me^{-j\beta x}}{2\pi\omega\mu_o\left(\epsilon-\left(\frac{\beta}{k}\right)^2\right)\left(1-\left(\frac{\beta}{k}\right)^2\right)}\int_{-\infty}^{\infty}\left[\frac{\epsilon\left(1-\left(\frac{\beta}{k}\right)^2\right)-\left(\frac{\beta}{k}\right)^2}{\varkappa_2+\epsilon\varkappa_1}+\frac{\left(\frac{\beta}{k}\right)^2}{\varkappa_2+\varkappa_1}\right]e^{-j\eta y}e^{-j\varkappa_1(z+z')}d\eta$$

$$\Pi' = \frac{M\frac{\beta}{k}}{2\pi k\left(\epsilon-\left(\frac{\beta}{k}\right)^2\right)\left(1-\left(\frac{\beta}{k}\right)^2\right.}\int_{-\infty}^{\infty}\frac{\eta}{\varkappa_1}\left[\frac{1}{\varkappa_1+\varkappa_2}-\frac{1}{\varkappa_2+\epsilon\varkappa_1}\right]e^{-j\eta y}e^{-j\varkappa_1(z+z')}d\eta \qquad \text{B(28)}$$

Appendix C

Determination of the sheets of κ_2 and κ_1 on which the Sommerfeld poles are located.

The poles of the integrand in (14) and (15) of G and H are given by

$$\kappa_2 + \epsilon \kappa_1 = 0 \qquad \text{C(1)}$$

Without loss of generality one may assume that the poles are located on the proper sheet $I_m \kappa_2 < 0$ of κ_2 (otherwise we may change signs of both square roots.) The question arises, are these poles located on the proper sheet of $\kappa_1 (I_m \kappa_1 < 0)$ or the improper sheet of κ_1. Equation C(1) may be written:

$$\sqrt{k^2 \epsilon - \alpha^2} + \epsilon \sqrt{k^2 - \alpha^2} = 0 \quad , \quad \alpha^2 = \eta^2 + \beta^2 \qquad \text{C(2)}$$

and are given by

$$\alpha_p^2 = k^2 \frac{\epsilon}{1+\epsilon} \quad . \qquad \text{C(3)}$$

Keeping in mind that $\epsilon_r > 0$, $\epsilon_i < 0$, we have

$$\kappa_2 = k \sqrt{\frac{\epsilon^2}{1+\epsilon}} \sim k \sqrt{\epsilon}$$

so that on the proper sheet of κ_2 $(\kappa_{2i} < 0)$, $\kappa_{2r} > 0$.

In a similar fashion, at the poles

$$\kappa_1 = \sqrt{k^2 - \frac{k^2 \epsilon}{1+\epsilon}} = k \sqrt{\frac{1}{1+\epsilon}} \qquad \text{C(4)}$$

Since with $\epsilon_r > 0$ and $\epsilon_i < 0$, $\frac{1}{1+\epsilon}$ is in the first quadrant, $\sqrt{\frac{1}{1+\epsilon}}$ is either in the first or the third quadrant of κ_1. The former corresponds to the

improper sheet $\kappa_{1i} > 0$ and since in the first quadrant, $\kappa_{1r} > 0$. The latter yields $\kappa_{1i} < 0, \kappa_{1r} < 0$. To determine which cases are possible let us rewrite C(1) in terms of its real and imaginary parts.

$$\kappa_{2r} + j\,\kappa_{2i} + (\epsilon_r + j\epsilon_i)(\kappa_{1r} + j\,\kappa_{1i}) = 0 \qquad C(5)$$

or

$$\left.\begin{aligned} &\kappa_{2r} + \epsilon_r\,\kappa_{1r} - \epsilon_i\,\kappa_{1i} = 0 \qquad C(6) \\ &\kappa_{2i} + \epsilon_r\,\kappa_{1i} + \epsilon_i\,\kappa_{1r} = 0 \qquad C(7) \end{aligned}\right\} \quad \text{simultaneously}$$

The following signs arise in C(6) on the improper sheet of κ_1

$$\begin{array}{ccccc} \kappa_{2r} & +\epsilon_r & \kappa_{1r} & -\epsilon_i & \kappa_{1i} \\ >0 & >0 & >0 & <0 & >0 \end{array} \qquad C(8)$$

Thus, all terms are of the same sign and cannot add up to zero on the improper sheet of κ_1.

On the proper sheet of κ_1 one has for the *LHS* of *C*(6)

$$\begin{array}{ccccc} \kappa_{2r} & +\epsilon_r & \kappa_{1r} & -\epsilon_i & \kappa_{1i} \\ >0 & >0 & <0 & <0 & <0 \end{array} \qquad C(9)$$

Thus, the first term is positive and the last two negative and balance to zero. To verify vanishing of LHS of C(7) on the proper sheet of κ_1 one has

$$\begin{array}{ccccc} \kappa_{2i} & +\epsilon_r & \kappa_{1i} & +\epsilon_i & \kappa_{1r} \\ <0 & >0 & <0 & <0 & <0 \end{array} \quad . \qquad C(10)$$

Thus, the first two terms are negative and the third positive. The above argument establishes the location of Sommerfeld poles simultaneously on the proper sheets of κ_2 and κ_1.

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